Finite Element Method Applied to Acoustics

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1 Introduction

The Finite Element Method (FEM) is a method for solving equations by using approximations of continuous quantities by a set of discrete ones, at discrete points, often displayed into a mesh or a grid. As the FEM can be adapted to problems of great complexity, it can incorporate material proprieties, anisotropy, boundary conditions, and unusual geometry, it is a extremely powerful tool in the solution of important problems, such as, heat transfer, fluid mechanics, mechanical systems, electromagnetic problems and acoustic problem, as will we will se later.

Formulation of problems with the FEM is based on the minimization of the total potential energy of the system via a variational principle.

2 Acoustic Theory

The wave propagation of finite amplitude may be described by a combination of four equations, which are: the equation of continuity, the equation of motion, the equation of sate and the equation of entropy.

2.1 The Equation of Continuity

The equation of continuity basely states that mass is neither destroyed nor generated. It is a connection between the fluid motion and its compression or expansion. In order to derive the continuity equation, consider a small rectangular parallelepiped volume element $dV = dxdydz$, which is fixed in space and through which elements of the fluid travel. If we consider $\overrightarrow{u}$ as the fluid velocity and ρ its density, than we may derive the net rate with which the mass within the volume increases. Considering the net influx only in the x direction, ρu_x is the amount of mass that cross the element's face projection into the $y \times z$ plane per unit of time. The mass increase inside the volume due to the influx in the x direction is equals the difference between the amount of mass that enters and the amount that leaves the volume, that is

$$[\rho u_x - (\rho u_x + \frac{\partial(\rho u_x)}{\partial x})]dydz = -\frac{\partial(\rho u_x)}{\partial x}dV \quad (1)$$

Similar equations are derived for the y and z components, so that the total influx must be

$$-(\frac{\partial(\rho u_x)}{\partial x} + \frac{\partial(\rho u_y)}{\partial y} + \frac{\partial(\rho u_z)}{\partial z})dV = -\nabla\cdot(\rho \mathbf{u})dV \quad (2)$$

The rate at which the mass increases in the volume is $(\partial\rho/\partial t)dV$. The net influx must equal the rate of increase,

$$\frac{\partial\rho}{\partial t} + \nabla\cdot(\rho \mathbf{u})dV = 0 \quad (3)$$

2.2 The Equation of Motion

The Navier-Stokes equations are a set of nonlinear partial differential equations that describe the flow of fluids such as liquids and gases. This equation models weather or the movement of air in the atmosphere, ocean currents, water flow in a pipe, as well as many other fluid flow phenomena.

This equation may be derived by considering the mass, momentum and energy balances for an infinitesimal control volume $dV = dxdydz$, which moves with the fluid and contains a mass dm of fluid. The net force $\overrightarrow{df}$ on the element will accelerate ir according to Newton's second law $\overrightarrow{df} = \overrightarrow{a} dm$. If we consider the absence of viscosity, then the Navier-Stokes equation will degenerate into the Euler equation.

The variables to be solved for are the velocity components, the fluid density, static pressure, and temperature. The flow is assumed to be differentiable and continuous, allowing these balances to be expressed as partial differential equations. The equations can be converted to Wilkinson equations for the secondary variables vorticity and stream function. The solution will depends on the fluids properties (such as viscosity, specific heats, and thermal conductivity), and on the boundary conditions of the domain of study.

The Navier-Stokes equation may be written as

$$\rho[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \bigtriangledown)\mathbf{u}] + \nabla p = \eta \Delta \mathbf{u} + (\frac{\eta}{3} + \zeta)\nabla(\nabla \cdot \mathbf{u}) \quad (4)$$

2.3 The Equation of Sate

For fluid media the equation of state must relates three physical quantities describing the thermodynamic behavior of the fluid. It gives the general relationship between the total pressure P(in pascals, Pa), the density ρ (in kilograms per cubic meter, kg/m^3), and the absolute temperature T_K (in kelvins, K), for a large number of gases under equilibrium conditions. This relation may be written as:

$$P = \rho r T_K \quad (5)$$

The constant r is the specific gas constant and depends on the universal gas constant R and the molecular weight M of the particular gas. For air $r \simeq 287J(kg \cdot K)$.

I acoustics processes the fluid behavior is almost isentropic, that means it is adiabatic and reversible. The thermal conductivity and the temperature gradient in the fluid are so small that no appreciable thermal energy transfer occurs in between the adjacent fluids elements. Under this assumption the entropy of the fluid remains nearly constant. Under these conditions, the acoustic behavior of the perfect gas is described by an adiabat

$$P/P_0 = (\rho/\rho_0)^\gamma \quad (6)$$

where γ is the ratio of specific heats. Finite thermal conductivity results in a conversion of acoustic energy into random thermal energy so that the acoustic disturbance attenuates slowly with time or distance.

For non-perfect gases, the adiabt is more complicated. In this situation it is preferable to determine the isentropic relationship between pressure and density fluctuations experimentally. This relationship can be expressed as a Taylor's expansion

$$P = P_0 + (\frac{\partial P}{\partial \rho})_{\rho_0}(\rho - rho_0) + \frac{1}{2}(\frac{\partial^2 P}{\partial \rho^2})_{\rho_0}(\rho - \rho_0)^2 + \ldots \quad (7)$$

$$\ldots \quad (8)$$

$$p = c_0^2 \rho + \frac{c_0^2}{\rho_0}\frac{B}{2A}\rho^2 + (\frac{\partial P}{\partial s})_{\rho,0} s \quad (9)$$

2.4 The Equation of Entropy

$$\rho T[\frac{\partial s}{\partial t} + (\mathbf{u} \cdot \nabla)s)] = \quad (10)$$

$$\kappa \nabla^2 T + \zeta(\nabla \cdot \mathbf{u})^2 + \frac{1}{2}\eta(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\delta_{ij}\frac{\partial U_k}{x_k})^2 \quad (11)$$

2.5 Combining Equations

Combining the four equations that describes the wave propagation and introducing the scalar velocity potencial ψ, we that the non-linear wave equation

$$\Delta\psi - \frac{1}{c_0^2}\frac{\partial^2 \psi}{\partial t^2} = -\frac{1}{c_0^2}\frac{\partial}{\partial t}(b(\Delta\varphi) + \frac{B/A}{2c_0^2}(\frac{\partial \psi}{\partial t})^2 + (\Delta\psi)^2) \quad (12)$$

The scalar potential velocity is defined by the following relations to the acoustic pressure and velocity

$$p = \rho_0 \frac{\partial \psi}{\partial t} \quad (13)$$

and

$$\mathbf{u} = -\nabla\psi \tag{14}$$

3 Solving Differential Equations with FEM

In order to solve differential equations will establish its original formulation, and we shall call it, the strong form of the problem (S). Then we will establish its variational form, or weak form (W), and than the Galerkin approximation of the weak form (G), which will lead us to a system of equations that might be represented using a matrix notation.

3.1 Strong Form

Let's suppose we want to solve a simple differential equation

$$u_{,xx} - f = 0 \tag{15}$$

where $u_{,xx}$ stands for the second derivative of u, that is, $u_{,xx} = \partial^2 u/\partial t^2$. Let's also suppose that f is a scalar and smooth function on the domain Ω under interest. For simplicity we shall consider Ω as the interval $[0,1]$. So, f is $f : [0,1] \rightarrow I\!R$

We also want that u satisfies some boundary conditions. Let's suppose they are

$$u(1) = g \tag{16}$$

$$-u_{,x}(0) = h \tag{17}$$

where g and h are both constants.

We may state the strong formulation of the problem as

Given $f : \Omega \rightarrow I\!R$ and constants g and h, find $u : \Omega \rightarrow I\!R$ such that

$$u_{,xx} - f = 0 u(1) = g - u_{,x}(0) = h \tag{18}$$

The analytic solution of this problem is simple and is

$$u(x) = g + (1-x)h + \int_x^1 (\int_0^y f(z)dz)dy \tag{19}$$

3.2 Weak Form

In order to define the variational form of the problem, we need to define to classes of function, the test functions and the weighting functions.

Test functions are functions that are possible solutions, so we might require them to satisfy the boundary condition $u(1) = g$. We shall also require those functions to have squared integrable derivatives, so that further expressions make sense, so

$$\int_0^1 (u_{,x})^2 dx < \infty \tag{20}$$

Functions that satisfies this propriety are called Hilber-1 functions, $u \in H^1$. The set of test functions is defined by

$$S = \{u/u \in H^1, u(1) = g\} \tag{21}$$

The set of weighting functions is quite similar to the test functions, but it requires the homogeneous counterpart of the boundary condition in $t = 1$, we require that the weighting functions w satisfies $w(1) = 0$. The set of weighting functions is defined by

$$V = \{w/w \in H^1, w(1) = 0\} \tag{22}$$

We may now take equation 15 and multiply it by a weighting function

$$w(u_{,xx} - f) = 0 \tag{23}$$

This must hold for any weighting function. If we integrate in the domain under concern, we will get the residual form

$$\int_\Omega w(u_{,xx} - f) = 0 \tag{24}$$

We may than integrate it by parts and obtain

$$\int_0^1 w_{,x} u_{,x} dx = \int_0^1 w f dx + w(0)h \tag{25}$$

We are going to use the following notation

$$a(w,u) = \int_0^1 w_{,x} u_{,x} dx \tag{26}$$

$$(w,f)=\int_0^1 wf dx \quad (27)$$

So the weak formulation of the problem (W) is

Given f, g and h, as before. Find $u \in S$ such that for all $w \in V$

$$a(w,u)=(w,f)+w(0)h \quad (28)$$

3.3 Galerkin Approximations

The Galerkin approximation is used to obtain an approximate solution of the boundary value problem, using the variational formulation.

The first step in developing this method is obtain finite dimensional approximation to the spaces of functions S and V, that we shall call S^h and V^h. The superscript h refers to the association of the spaces S^h and V^h to mesh (or the discretization of the domain), which is has a characteristic length h. S^h and V^h are sub-conjuncts of S and V, that is, $S^h \subseteq S$ and $V^h \subseteq V$, so the functions in S^h and V^h satisfy the same conditions that the functions of S and V.

For a each $v^h \in V^h$ we may create a function $u^h \in S^h$ where

$$u^h = v^h + g^h \quad (29)$$

where g^h is a function that satisfies the boundary condition

$$g^h(1) = g \quad (30)$$

so u^h shall also satisfies this boundary condition, since $v^h(1)=0$.

Using the Galerkin approximation we may rewrite the variational formulation as

$$a(w^h,u^h)=(w^h,f)+w^h(0)h \quad (31)$$

Using 29 we have

$$a(w^h,v^h)=(w^h,f)+w^h(0)h-a(w^h,g^h) \quad (32)$$

The Galerkin form of the problem is

Given f, g and h, as before. Find $u^h=v^h+g^h$, where $v^h \in V^h$, such that for all $w^h \in V^h$

$$a(w^h,v^h)=(w^h,f)+w^h(0)h-a(w^h,g^h) \quad (33)$$

3.4 Matrix Equation

Gallerkin method lead us to a linear coupled system of algebraic equations.

If we deal V^h as a vectorial space, as it is, we may say that every function, vector, in it is a linear weighted combination of the base functions $N_A : \Omega \rightarrow I\!R$, $A=1,2,...,n$. Any vector in V^h may be written as

$$w^h=\sum_{A=1}^{n} c_A N_A \quad (34)$$

The functions N_A are also called form functions or interpolation functions. So we will require that those functions satisfies

$$N_A(1)=0, \qquad A=1,2,...,n \quad (35)$$

From this we have w^h satisfying $w^h(1)=0$, as it was necessary, so that w^h belongs to V^h.

In order to define the member functions of S^h, it is necessary to define g^h. We will then introduce another form function $N_{n+1}:\Omega \rightarrow I\!R$ with the following propriety:

$$N_{n+1}(1)=1 \quad (36)$$

N_{n+1} is not a member of V^h. g^h is given by

$$g^h = gN_{n+1} \quad (37)$$

so, $g^h(1)=g$.

Using this definition, a typical vector $u^h \in S^h$ is

$$u^h=v^h+g^h \quad (38)$$

$$u^h=\sum_{A=1}^{n} d_A N_A + gN_{n+1} \quad (39)$$

It's easy to verify that $h^h(1)=g$.

Using the above expressions 34 and 39 into the Galerkin approximation 33

$$a(\sum_{A=1}^{n} c_A N_A, \sum_{B=1}^{n} d_B N_B) = \quad (40)$$

$$(\sum_{A=1}^{n} c_A N_A, f)+[\sum_{A=1}^{n} c_A N_A(0)]h - a(\sum_{A=1}^{n} c_A N_A, gN_{n+1}) \quad (41)$$

as the operators $a(.,.)$ and $(.,.)$ are bilinear operators, the above expression may be rewritten as

$$0 = \sum_{A=1}^{n} c_A G_A \qquad (42)$$

where

$$G_A = \sum_{B=1}^{n} a(N_A, N_B) d_B - (N_A, f) - \qquad (43)$$

$$N_A(0)h + a(N_A, N_{n+1})g \qquad (44)$$

The Galerkin approximation must hold for every $w^h \in V^h$. So, the equation 42 is valid for all coefficients c_A's, $A = 1, 2, ..., n$. As those coefficients are arbitrary, 42 is valid only if G_A is null for $A = 1, 2, ..., n$. In this way we have

$$\sum_{B=1}^{n} a(N_A, N_B) d_B = (N_A, f) + N_A(0)h - a(N_A, N_{n+1})g \qquad (45)$$

All the terms above are know, but the coefficients d_B's. We have though a system of n equations and n variables.

Using the notation

$$K_{AB} = a(N_A, N_B) \qquad (46)$$

$$F_A = (N_A, f) + N_A(0)h - a(N_A, N_{n+1})g \qquad (47)$$

we have

$$\sum_{B=1}^{n} K_{AB} d_B = F_A, A = 1, 2, ..., n \qquad (48)$$

That we may write in a matrix form

$$\mathbf{K} = [K_{AB}] = \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & K_{22} & \dots & K_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ K_{n1} & K_{n2} & \dots & K_{nn} \end{bmatrix} \qquad (49)$$

where $\mathbf{K}$ is symmetric due to the bilinear propriety of $a(.,.)$.

$$\mathbf{F} = [F_A] = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{bmatrix} \qquad (50)$$

and

$$\mathbf{d} = [d_B] = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix} \qquad (51)$$

The system becomes

$$\mathbf{Kd} = \mathbf{F} \qquad (52)$$

$\mathbf{K}$ is known as stiffness matrix, $\mathbf{F}$ as the forcing vector and $\mathbf{d}$ as the displacement vector.

The problem now has become, to find the displacement vector $\mathbf{d}$ given the coefficient matrix $\mathbf{K}$ and the forcing vector $\mathbf{F}$.

The system solution is obvious, $\mathbf{d} = \mathbf{K^{-1}F}$ (if such a inverse exists). So we may now obtain the solution to the Galerkin approximation

$$u^h(x) = \sum_{A=1}^{n} d_A N_A(x) + g N_{n+1}(x) \qquad (53)$$

This solution is a approximations to the weak formulation solution, consequently the differential equation and the boundary conditions are weakly satisfied. The quality of the approximation depends on the choice of the base functions N_A's and also on the number of function used.

Referências

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